

The Representation of Line Dirac Delta Function Along a Space Curve

Zhou Zhang*

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

Xiaoming Zheng†

Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859

Abstract

In this paper, we describe the line Dirac delta function of a curve in three-dimensional space in terms of the distance function to the curve. Its extension to level set formulation and plane curves are also developed.

Keyword. line Dirac delta function, distance function, level set function

1 Introduction

A line source or sink is often encountered in electromagnetic and biological problems when a long and thin structure is involved. For instance, a distribution of electric charge on a curve can be represented by a line source [1]. A flexible structure such as an elastic fiber or filament is modeled as a line source in Immersed Boundary method to address the mechanical interactions between the structure and biofluid [7, 5]. A very thin blood capillary in a bulk tumor can be treated as a line source when modeling a growth factor's reaction and diffusion [8]. In these models, the line source is considered as a simple smooth curve Σ , embedded in the three-dimensional (3-D) Euclidean space \mathbb{R}^3 or its generalization.

Denote the arc length parameter by s and the corresponding spatial point on this curve by $\mathbf{x}(s)$. The line delta function δ_Σ associated with the curve Σ is defined as a distribution such that for any test function $f(\mathbf{x}) \in C_c^\infty(\mathbb{R}^3)$ (i.e., infinitely differentiable and compactly supported),

$$\int_{\mathbb{R}^3} \delta_\Sigma(\mathbf{x}) \cdot f(\mathbf{x}) d\mathbf{x} = \int_\Sigma f(\mathbf{x}(s)) ds. \quad (1)$$

In Cartesian coordinates, by formally switching the order of integrations (i.e. applying “Fubini’s Theorem”), it can be understood as $\delta_\Sigma(\mathbf{x}) = \int_\Sigma \delta^{3D}(\mathbf{x} - \mathbf{x}(s)) ds, \forall \mathbf{x} \in \mathbb{R}^3$, where $\delta^{3D}(\mathbf{x}) =$

*zhangou@maths.usyd.edu.au

†zhenglx@cmich.edu.

$\delta(x)\delta(y)\delta(z)$ for $\mathbf{x} = (x, y, z)$, where $\delta(x)$ is the one-dimensional (1-D) Dirac delta function satisfying $\int_{-\infty}^{\infty} \delta(t)g(t)dt = g(0)$ for any $g \in C_c^\infty(\mathbb{R})$.

The purpose of this work is to provide an alternative representation of the line delta function δ_Σ for a fairly general class of spatial curves sufficient for most practical applications, which is

$$\delta_\Sigma(\mathbf{x}) = \frac{\delta(\rho(\mathbf{x}))}{2\pi\rho(\mathbf{x})}, \quad (2)$$

where $\rho(\mathbf{x})$ is the distance function to the curve. This δ function is under the convention that $\int_0^\infty \delta(t)g(t)dt = g(0)$ for any $g \in C_c^\infty(\mathbb{R})$.

The main challenge is to how to make sense of and calculate $\int_{\mathbb{R}^3} \frac{\delta(\rho)}{2\pi\rho} f(\mathbf{x})d\mathbf{x}$ for any $f \in C_c^\infty(\mathbb{R}^3)$. For this, one has to choose the most reasonable and practical coordinate system. Indeed, the apparently singular term $\frac{1}{\rho}$ on the right hand side comes naturally from the local cylindrical coordinates in a tubular neighbourhood near the curve. If the curve Σ is simple and smooth, then a tubular neighbourhood can be readily chosen where the distance function works as the radius and will be cancelled out in coordinate transformation. This will be done in Section 2. In general, if the curve has non-smooth points, we have to make the meaning of $\frac{\delta(\rho)}{2\pi\rho}$ more clear, which will be done in Section 3. As application, the formula (2) works for very complicated structure composed by piecewise smooth curves such as the newly formed blood capillary network in growing tumor, as shown in Fig. 1.

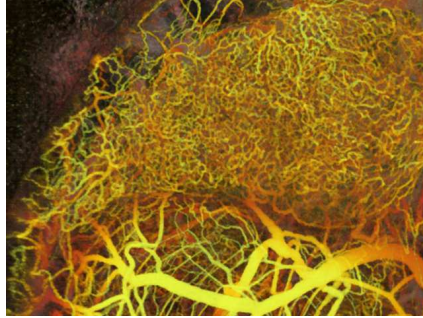


Figure 1: The highly irregular and tortuous blood vessel capillaries in a xenotransplanted U87 human glioblastoma multiforme tumor (upper part) in a mouse brain. The size of tissue shown in this figure is 2.6 mm by 2 mm . This picture is taken from [6] with permission.

Another important application of this work is in the level set method. The delta function of a curve or a surface represented by a level set function is crucial in the level set method [4], but the form for a spatial curve is not previously available. The formula (2) provides the first representation with the distance function, whose generalization to an arbitrary level set function will be given in Section 4. The extension to plane curves and general submanifolds with co-dimension 1 and 2 are also discussed in Section 4.

2 Case I: simple smooth curves embedded in \mathbb{R}^3

The essential tool in our calculation of $\frac{\delta(\rho)}{2\pi\rho}$ is the knowledge of a neighbourhood around the curve. The standard tubular neighbourhood theorem is for compact smooth submanifolds of \mathbb{R}^k (c.f.

[2]). Therefore, when it is applied to a curve, the curve needs to be a submanifold of \mathbb{R}^3 and is diffeomorphic to a circle. However, because of the local nature of our problem, it can be readily applied to any simple smooth curve which is a submanifold of \mathbb{R}^3 . More detail is as follows.

Let Σ be a simple smooth curve with the arc length parameter s , any point on the curve denoted as $\mathbf{x}(s)$. Look at a finite piece of it with $s \in [s_1, s_2]$, for simplicity, still denoted as Σ . It is a closed set in \mathbb{R}^3 with the two endpoints $\mathbf{x}(s_1)$ and $\mathbf{x}(s_2)$. Denote $\mathring{\Sigma}$ as the interior part by removing the two endpoints of the curve Σ . On each point $\mathbf{x}(s) \in \mathring{\Sigma}$, denote the tangent space as $T_{\mathbf{x}(s)}\Sigma$ which is exerted by $\frac{\partial}{\partial s}$. Define the normal plane as $N_{\mathbf{x}(s)}\Sigma = \{\mathbf{z} \in \mathbb{R}^3 | \mathbf{z} \perp T_{\mathbf{x}(s)}\Sigma\}$. On the two endpoints $\mathbf{x}(s_1)$ and $\mathbf{x}(s_2)$, define tangent and normal planes as the limits of those of points in $\mathring{\Sigma}$. Let $B(\mathbf{x}(s_1), \epsilon)$ be the open ball centered at $\mathbf{x}(s_1)$ with radius ϵ , and $B^{out}(\mathbf{x}(s_1), \epsilon)$ be the open semi-ball of $B(\mathbf{x}(s_1), \epsilon)$ cut by the normal plane $N_{\mathbf{x}(s_1)}\Sigma$ which is disjoint with $\mathring{\Sigma}$. Similarly we can define $B^{out}(\mathbf{x}(s_2), \epsilon)$.

Define the tubular neighbourhood

$$\text{Tub}^\epsilon(\Sigma) = \{\mathbf{x} \in \mathbb{R}^3 | \text{dist}(\mathbf{x}, \Sigma) < \epsilon\} \setminus \{B^{out}(\mathbf{x}(s_1), \epsilon) \cup B^{out}(\mathbf{x}(s_2), \epsilon)\},$$

which is the set of all points within distance ϵ to Σ but excluding the two outside semi-balls at the two ends of the curve. Note $N_{\mathbf{x}(s_1)}\Sigma$ and $N_{\mathbf{x}(s_2)}\Sigma$ are two bounding normal planes of this tube (see Fig. 2). Also define

$$\Xi(\Sigma, \epsilon) = \{(\mathbf{x}, \mathbf{z}) \in \Sigma \times \mathbb{R}^3 | \mathbf{z} \in N_{\mathbf{x}(s)}\Sigma, \|\mathbf{z}\| < \epsilon\}.$$

Let $\theta : \Xi(\Sigma, \epsilon) \rightarrow \mathbb{R}^3$ be given by $\theta(\mathbf{x}, \mathbf{z}) = \mathbf{x} + \mathbf{z}$. The following is a standard result in differential geometry.

Lemma 1. *There exists $\epsilon > 0$ such that the map θ is a diffeomorphism from $\Xi(\Sigma, \epsilon)$ onto $\text{Tub}^\epsilon\Sigma$.*

Remark 1. *A prominent feature of this tubular neighbourhood is that $\forall \mathbf{y} \in \text{Tub}^\epsilon\Sigma$, $\text{dist}(\mathbf{y}, \Sigma) = \|\mathbf{z}\|$ where the vector \mathbf{z} is from the unique decomposition $\mathbf{y} = \mathbf{x} + \mathbf{z}$, $(\mathbf{x}, \mathbf{z}) \in \Xi(\Sigma, \epsilon)$. Also notice that for our curve of infinite length, one might not have a uniform ϵ for the whole curve, but this is not going to give us any trouble in this work because only pieces of finite length will be considered in the proof of Theorem 1.*

Define the distance function to Σ as $\rho(\mathbf{x}) \triangleq \inf_{\mathbf{y} \in \Sigma} d(\mathbf{x}, \mathbf{y})$, where $d(\mathbf{x}, \mathbf{y})$ is the standard Euclidean distance. $\rho(\mathbf{x}) = 0$ implies $\mathbf{x} \in \Sigma$ because the curve Σ is embedded in \mathbb{R}^3 (i.e. being a submanifold).

Theorem 1. *Let Σ be a simple smooth curve embedded in \mathbb{R}^3 , then*

$$\delta_\Sigma(\mathbf{x}) = \frac{\delta(\rho)}{2\pi\rho}. \quad (3)$$

Proof. Around any point on Σ , we have a tubular neighbourhood $\text{Tub}^\epsilon\Sigma$ satisfying the condition in Lemma 1. First, a coordinate system will be constructed in $\text{Tub}^\epsilon\Sigma$ as follows. On the curve Σ , $\frac{\partial}{\partial s}$ is a unit vector field. Because the curve is smooth, there exist two other smooth unit vector fields U and V such that U , V , and $\frac{\partial}{\partial s}$ form an orthonormal system and satisfy the right hand rule. On each normal plane $N_{\mathbf{x}(s)}\Sigma$, U and V provide a \mathbb{R}^2 coordinate system, (u, v) . Denote any point on this plane as $\mathbf{x}(u, v, s)$ (see Fig. 2). According to Remark 1, $\rho(\mathbf{x}(u, v, s)) = \sqrt{u^2 + v^2}$.

Because $\{U, V, \frac{\partial}{\partial s}\}$ is a smooth frame along the curve, the map from (u, v, s) to (x, y, z) is smooth. The Jacobian between (x, y, z) and (u, v, s) is the determinant of the transition matrix

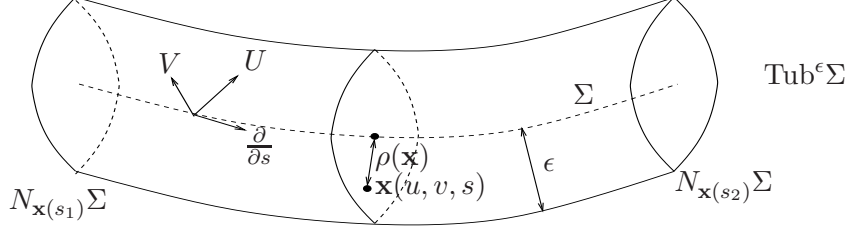


Figure 2: A tubular neighbourhood $\text{Tub}^\epsilon \Sigma$ of a simple smooth curve Σ in 3-D space bounded by two normal planes $N_{\mathbf{x}(s_1)} \Sigma$ and $N_{\mathbf{x}(s_2)} \Sigma$.

between the \mathbb{R}^3 standard basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ and $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial s}\}$ which is just $\{U, V, \frac{\partial}{\partial s}\}$ when restricted on the curve. Although $\frac{\partial}{\partial s}$ is not explicit away from the curve, the transition matrix has to be in $SO(3)$ on the curve because it is between two orthonormal bases with the same (right-handed) orientation. So the determinant is 1 on the curve. As the Jacobian is smooth in the tubular neighbourhood $\text{Tub}^\epsilon \Sigma$, it's $1 + O(\sqrt{u^2 + v^2})$ in sight of the Taylor series. Thus the Jacobian must be nonzero in a sufficiently small neighbourhood, say, $\text{Tub}^{\epsilon_1} \Sigma, 0 < \epsilon_1 < \epsilon$. For simplicity, it is still denoted as $\text{Tub}^\epsilon \Sigma$. Therefore, (u, v, s) is a coordinate system in $\text{Tub}^\epsilon \Sigma$.

When converted to the polar version (ρ, θ, s) , since $\det \left(\frac{\partial(u, v, s)}{\partial(\rho, \theta, s)} \right) = \rho$, it is easy to see

$$\det \left(\frac{\partial(x, y, z)}{\partial(\rho, \theta, s)} \right) = \rho + O(\rho^2) \text{ in } \text{Tub}^\epsilon \Sigma.$$

For any test function $f(\mathbf{x}) \in C_c^\infty(\mathbb{R}^3)$, $\int_{\mathbb{R}^3} \frac{\delta(\rho)}{2\pi\rho} f(\mathbf{x}) d\mathbf{x}$ only depends on the situation near Σ as $\delta(\rho)$ is supported on Σ . Since f is compactly supported, we can assume that it is supported in a ball B . Because $B \cap \Sigma$ is compact, there exist finitely many local tubular neighbourhoods centered at involved pieces of Σ covering $B \cap \Sigma$, denoted as $\{T_i\}_{i \in I}$. We can also make sure that $\{\hat{T}_i\}$ with $\hat{T}_i \Subset T_i$ also covers $B \cap \Sigma$. Now a standard partition of unity construction provides us with $h_i \in C_c^\infty(T_i)$ such that $\sum_{i \in I} h_i = 1$ when restricted to $\cup_{i \in I} \hat{T}_i$. Finally, we can carry out the following computation.

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{\delta(\rho)}{2\pi\rho} f(\mathbf{x}) d\mathbf{x} &= \int_{\mathbb{R}^3} \sum_{i \in I} \frac{\delta(\rho)}{2\pi\rho} f(\mathbf{x}) h_i(\mathbf{x}) d\mathbf{x} = \sum_{i \in I} \int_{\hat{T}_i} \frac{\delta(\rho)}{2\pi\rho} f(\mathbf{x}) h_i(\mathbf{x}) d\mathbf{x} \\ &= \sum_{i \in I} \int_s \int_\theta \int_\rho \frac{\delta(\rho)}{2\pi\rho} f(\rho, \theta, s) h_i(\rho, \theta, s) (\rho + O(\rho^2)) d\rho d\theta ds \\ &= \sum_{i \in I} \int_s \int_\theta \int_\rho \frac{\delta(\rho)}{2\pi} f(\rho, \theta, s) h_i(\rho, \theta, s) (1 + O(\rho)) d\rho d\theta ds \\ &= \sum_{i \in I} \int_s \int_0^{2\pi} \frac{1}{2\pi} f(0, \theta, s) h_i(0, \theta, s) d\theta ds \\ &= \int_s f(\mathbf{x}(s)) \sum_{i \in I} h_i(\mathbf{x}(s)) ds = \int_s f(\mathbf{x}(s)) ds = \int_\Sigma f(\mathbf{x}(s)) ds. \end{aligned}$$

In sight of the definition in Eq. (1), the theorem is proven.

□

Remark 2. *The choice of coordinates does not have to be fixed and is actually flexible. However, we do need to start with a Euclidean orthonormal basis along Σ with one vector in the direction of the curve. In short, the delta function would function on the normal plane of the curve with the Euclidean structure induced from \mathbb{R}^3 .*

3 Case II: general curves embedded in \mathbb{R}^3

With a proper understanding of the delta function, the previous argument can be adjusted to a general class of curves which are topological graphs with smooth edges. The graph can have infinite vertices with finite edges between any two of them and the edges need not be straight (i.e. being topological). We only need to require that there is no local “accumulation” happening, which is in general what people think of graphs.

Let Σ be such a curve. The whole curve Σ may be closed or open, may have non-smooth points and self-intersections, and may have more than one connected components (See Fig. 3). Denote the

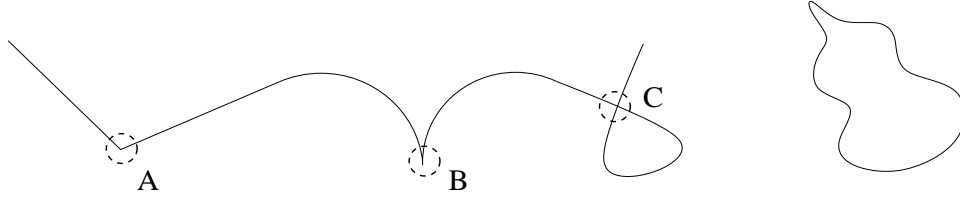


Figure 3: This curve has two connected components: the left component is non-closed, has non-smooth points at A and B, and crosses itself at C; the right component is closed and smooth.

collection of all non-smooth points as $P_\Sigma = \{\mathbf{p}_j, j \in \Gamma\}$, where Γ is a finite or countable index set. To handle these non-smooth points, we remove from Σ a small neighbourhood in \mathbb{R}^3 around each such point such that the boundary of the neighbourhood is perpendicular to the curve at the point of intersection. Denote the union of these neighbourhoods by U_ϵ and shrink U_ϵ to P_Σ as $\epsilon \rightarrow 0$. Let χ_ϵ be the characteristic function of $\mathbb{R}^3 \setminus U_\epsilon$, i.e., $\chi_\epsilon = 0$ over U_ϵ and 1 otherwise.

The following definition is what we want:

$$\frac{\delta(\rho)}{2\pi\rho} = \lim_{\epsilon \rightarrow 0} \frac{\chi_\epsilon \cdot \delta(\rho)}{2\pi\rho}. \quad (4)$$

Notice that $\frac{\chi_\epsilon \cdot \delta(\rho)}{2\pi\rho}$ can be understood in the same way as before because a neighbourhood of the non-smooth points has essentially been removed. The limit is in the weak sense, i.e. in the sense of current, with the existence justified by the following computation. For any $f \in C_c^\infty(\mathbb{R}^3)$,

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} \frac{\chi_\epsilon \cdot \delta(\rho)}{2\pi\rho} f(\mathbf{x}) d\mathbf{x} = \lim_{\epsilon \rightarrow 0} \int_\Sigma \chi_\epsilon \cdot f(\mathbf{x}(s)) ds = \int_\Sigma f(\mathbf{x}(s)) ds$$

The first equality makes use of the same kind of computation as that in the previous section, i.e., using a partition of unity construction to reduce to local tubular neighbourhoods with cylindrical coordinates. This is done for each ϵ and the local tubular neighbourhoods only need to cover $\Sigma \setminus U_\epsilon$.

This also shows the independence of the choice (on χ_ϵ) in the construction. Simply speaking, there is no contribution from P_Σ in integration. In summary, we have proven the following main theorem.

Theorem 2. *Let Σ be a topological graph in \mathbb{R}^3 with smooth edges and ρ be the distance function to Σ . Then the definition (4) makes sense and $\delta_\Sigma(\mathbf{x}) = \frac{\delta(\rho)}{2\pi\rho}$.*

4 Extensions: level set function, plane curves, and more

Let ϕ be a non-negative level set function of Σ , that is, $\Sigma = \{\mathbf{x} \mid \phi(\mathbf{x}) = 0\}$. Assume ϕ is a function, $\phi = \phi(\rho, \theta, s)$, locally near Σ in the coordinate system used before, and $\frac{\partial\phi}{\partial\rho} \neq 0$ around Σ . Since ϕ is non-negative, we have actually assumed $\frac{\partial\phi}{\partial\rho} > 0$. This implies (ϕ, θ, s) can be viewed as a generalized cylindrical coordinate system locally near Σ , and the Jacobian between it and (ρ, θ, s) is $\frac{\partial(\phi, \theta, s)}{\partial(\rho, \theta, s)} = \frac{\partial\phi}{\partial\rho}$. Therefore, we have the following result.

Theorem 3. *Let Σ be a topological graph in \mathbb{R}^3 with smooth edges, ρ be the distance function to Σ , and $\phi(\mathbf{x})$ be a non-negative level set function of Σ as above, then $\delta_\Sigma(\mathbf{x}) = \frac{\delta(\phi)\frac{\partial\phi}{\partial\rho}}{2\pi\rho}$.*

The key is that $d\rho d\theta ds = \frac{1}{\partial\phi/\partial\rho} d\phi d\theta ds$. For non-smooth points, one can have similar discussion as in Section 3.

Analogous to Theorem 2 and 3, we can obtain the following results for plane curves.

Theorem 4. *Let Σ be a topological graph in \mathbb{R}^2 with smooth edges, ρ be the distance function to Σ , and $\phi(\mathbf{x})$ be a non-negative level set function of Σ , with $\frac{\partial\phi}{\partial\rho}$ is uniformly bounded away from 0, then $\delta_\Sigma(\mathbf{x}) = \delta(\rho) = \delta(\phi)|\nabla\phi|$.*

Proof. We only need to prove the level set formulation for a simple smooth curve Σ embedded in \mathbb{R}^2 . We use the 2-D coordinates (ρ, s) in the tubular neighbourhood U around Σ . Over Σ , $\nabla\phi = \hat{\rho}\frac{\partial\phi}{\partial\rho} + \hat{s}\frac{\partial\phi}{\partial s} = \hat{\rho}\frac{\partial\phi}{\partial\rho}$ as $\frac{\partial\phi}{\partial s} = 0$ over Σ , where $\hat{\rho}$ and \hat{s} are orthonormal vectors in the direction of $\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial s}$, respectively. Then over Σ (i.e. $\{\mathbf{x} \mid \rho(\mathbf{x}) = 0\} = \{\mathbf{x} \mid \phi(\mathbf{x}) = 0\}$), $|\nabla\phi| = \frac{\partial\phi}{\partial\rho} > 0$. Take any test function $f \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned} \int_{\mathbb{R}^2} f(\mathbf{x})\delta(\phi)|\nabla\phi|d\mathbf{x} &= \int_U f(\mathbf{x})\delta(\phi)|\nabla\phi|Jd\rho ds \\ &= \int_U f(\mathbf{x})\delta(\phi)|\nabla\phi|J\frac{1}{\partial\phi/\partial\rho}d\phi ds \\ &= \int_\Sigma f(\mathbf{x}(s))ds, \end{aligned}$$

where J is the Jacobian between the Euclidean coordinate and (ρ, s) , which is equal to 1 along Σ because $\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial s}$ form an orthonormal basis there. □

Remark 3. *For Theorem 3, $\frac{\partial\phi}{\partial\rho}$ is not well-defined as a function over Σ in general because of the presence of θ , and should be understood in the integration sense. Because we do not have θ for a neighbourhood of a curve in \mathbb{R}^2 , this is not an issue for Theorem 4.*

Remark 4. In the 3-D case, a succinct formula like $\frac{\delta(\phi)}{2\pi\rho}|\nabla\phi|$ does not hold in general. Indeed, in 3-D cylindrical coordinates, $|\nabla\phi| \sim \sqrt{\phi_\rho^2 + \frac{1}{\rho^2}\phi_\theta^2 + \phi_s^2}$ near Σ . Therefore, when approaching Σ , $\frac{\partial\phi}{\partial\rho}$ cannot be replaced by $|\nabla\phi|$ unless $\phi_\theta/\rho \rightarrow 0$, which puts restriction on this level set function ϕ .

Remark 5. The study here can be extended to the general case of a submanifold, Σ , of co-dimension 2 or 1 in \mathbb{R}^n with $n > 3$ (or any smooth Riemannian manifold in general). That is, $\delta_\Sigma = \frac{\delta(\rho)}{2\pi\rho}$ for a submanifold Σ with co-dimension 2, while $\delta_\Sigma = \delta(\rho) = \delta(\phi)|\nabla\phi|$ if Σ is of co-dimension 1.

Indeed, the form $\delta_\Sigma = \delta(\phi)|\nabla\phi|$ for smooth closed curves in \mathbb{R}^2 and surfaces in \mathbb{R}^3 was first derived in [3] and now is widely used in the level set method. Hence, the results presented in this work can be regarded as extensions of [3].

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